



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII.

NOVEMBER, 1906.

No. 11

ON LINEAR ALGEBRAS.

By DR. L. E. DICKSON.

In the theory of algebraic numbers we consider such systems of numbers as the set of all numbers $r+si+ti^2$, in which r, s, t are rational, while i is an irrational number satisfying an equation $x^3-\beta x-b=0$ with rational coefficients. This set of numbers is called a field (or domain) since the sum, difference, product, or quotient, of any two of the numbers is likewise a unique number of the set. If we put $i^2=j$, we may consider this field to be a linear algebra composed of all numbers $r+si+tj$ (r, s, t rational), in which the "units" $1, i, j$ satisfy the relations

$$\begin{aligned} (1) \quad & 1^2=1, 1.i=i.1=i, 1.j=j.1=j, \\ (2) \quad & i^2=j, i.j=j.i=b+\beta i, j^2=bi+\beta j. \end{aligned}$$

In general, a linear triple algebra is determined by the "multiplication table" of its units [(1) and (2) in the above case] and by the field over which the coördinates r, s, t range [the field of rational numbers in the above case]. There is a very extensive literature on linear *associative* algebras, in which $i(jk)=(ij)k$ for any three units (not necessarily distinct). But only in rare instances, such as for quaternions, is division always possible; while then multiplication is not commutative. Another class of highly interesting linear algebras has been considered recently.* In these, multiplication is commutative and distributive, but not always associative, while *division is always uniquely possible*. We shall here

*Dickson, *Goettingen Nachrichten*, 1905, pp. 358—393; *Transactions American Mathematical Society*, Vol. 7 (1906), pp. 370—390; 514—522.

investigate such triple algebras by a new, simple method. Let* the units 1, i , j satisfy relations (1) and

$$(3) \quad i^2=j, \quad ij=ji=b+\beta i, \quad j^2=d+\delta i+Dj,$$

$x^3-\beta x-b$ being irreducible in the reference field F . Then

$$(4) \quad (r+si+tj)(x+yi+zj)=P+Qi+Rj,$$

where P , Q , R are given (by detached coefficients) by

$$(5) \quad \begin{array}{l} P= \\ Q= \\ R= \end{array} \begin{array}{|c|cc|} \hline & x & y & z \\ \hline r & tb & sb+td \\ s & r+t\beta & s\beta+td \\ t & s & r+tD \\ \hline \end{array}$$

The determinant (of the third order) of this array equals

$$(6) \quad \Delta = r^3 + r^2 t(\beta + D) + rt^2(\beta D - d) - rs^2\beta - rst(\delta + 2b) + s^3b + s^2td - st^2bD + t^3(b\delta - d\beta).$$

Uniqueness of division requires that, for given values of r , s , t (not all zero) and P , Q , R , there shall exist unique solutions x , y , z of (4) and hence of (5). *Hence division is always uniquely possible if and only if $\Delta=0$ implies $r=s=t=0$.* We thus require that the Hessian H of the cubic form Δ shall differ from Δ only by a constant factor. By definition, this Hessian is

$$H = \begin{vmatrix} \Delta_{rr} & \Delta_{rs} & \Delta_{rt} \\ \Delta_{rs} & \Delta_{ss} & \Delta_{st} \\ \Delta_{rt} & \Delta_{st} & \Delta_{tt} \end{vmatrix}, \quad \Delta_{rr} \equiv \frac{\partial^2 \Delta}{\partial r^2}, \quad \Delta_{rs} \equiv \frac{\partial^2 \Delta}{\partial r \partial s}, \text{ etc.}$$

For the cubic form Δ , given by (6),

$$\begin{aligned} \Delta_{rr} &= 6r + 2t(\beta + D), \quad \Delta_{rs} = -2s\beta - t(\delta + 2b), \quad \Delta_{rt} = 2r(\beta + D) + 2t(\beta D - d) \\ &\quad - s(\delta + 2b), \\ \Delta_{ss} &= -2r\beta + 6sb + 2td, \quad \Delta_{st} = -r(\delta + 2b) + 2sd - 2tbD, \quad \Delta_{tt} = 2r(\beta D - d) - 2sbD \\ &\quad + 6t(b\delta - d\beta). \end{aligned}$$

The cubic form H is found to have the coefficients

$$\begin{aligned} r^3 : A &= 8\beta(\beta + D)^2 - 6(\delta + 2b)^2 - 24\beta(\beta D - d), \\ s^3 : B &= 8\beta^2bD + 8\beta d(\delta + 2b) - 6b(\delta + 2b)^2, \\ t^3 : C &= [24d(\beta + D) - 6(\delta + 2b)^2] (b\delta - d\beta) + 8bD(\delta + 2b)(\beta D - d) - 8b^2D^2(\beta + D) \\ &\quad - 8d(\beta D - d)^2, \end{aligned}$$

*It is easily shown that there must occur a "modulus" 1, satisfying (1), and that not every element e satisfies a quadratic equation. Hence there exists an element i whose square is linearly independent of 1, i ; it may be taken as the third unit j .

$$\begin{aligned}
r^2s : J &= 24b\beta D + 72b(\beta D - d) + 24d(\delta + 2b) - 24b(\beta + D)^2, \\
s^2t : K &= 2d(\delta + 2b)^2 + [24b(\delta + 2b) - 16\beta d](\beta D - d) - 24\beta^2(d\delta - d\beta) \\
&\quad - (24b^2D + 8d^2)(\beta + D), \\
rs^2 : L &= -72b^2D - 24d^2 - 16\beta d(\beta + D) - 8\beta^2(\beta D - d) - 2\beta(\delta + 2b)^2 \\
&\quad + 24b(\beta + D)(\delta + 2b), \\
r^2t : M &= [24d + 8\beta(\beta + D)](\beta D - d) - 72\beta(b\delta - d\beta) - 24bD(\delta + 2b) \\
&\quad + 2(\beta + d)(\delta + 2b)^2 - 8d(\beta + D)^2,
\end{aligned}$$

the coefficients of rt^2 , st^2 and rst not being exhibited.

We require that the coefficients A , B , ... of the Hessian shall be proportional to the corresponding coefficients of Δ . It will not be necessary to examine all of these conditions. We exclude at present the special case in which the reference field F has a modulus* 2 or 3, so that we may divide by these numbers. Since there is no term r^2s in Δ , we have $J=0$,

$$(7) \quad 2b\beta D - b\beta^2 - bD^2 - bd + d\delta = 0.$$

The condition $A : B = 1 : b$ is then satisfied. From $C : A = (b\delta - d\beta) : 1$,

$$(8) \quad \beta^4d - \beta^3b\delta - \beta^3dD + \beta^2b\delta D - \beta Dd^2 + \beta b^2D^2 + 2\delta dDb - 2b^2dD - b^2D^3 - d^3 = 0.$$

From $K : A = d : 1$,

$$(9) \quad [d\beta + 3b(\delta + 2b)](\beta D - d) + d(\delta + 2b)^2 - \beta d(\beta + D)^2 - 3\beta^2(b\delta - d\beta) - (d^2 + 3b^2D)(\beta + D) = 0.$$

Now $b \neq 0$ in view of the assumed irreducibility of

$$(10) \quad x^3 - \beta x - b.$$

We assume here that $\beta \neq 0$. We may set

$$(11) \quad \delta = xb, \quad D = yb, \quad d = z\beta^2.$$

Then (7) has the factor $b\beta^2$, and we get

$$(12) \quad xz - z = (y - 1)^2.$$

Next, we multiply (8) and (9) by d , and eliminate $d\delta$ by means of (7). In the resulting conditions make the substitution (11). After suitable factorizations, we get

$$\begin{aligned}
(13) \quad & \beta^5(z+1)G = 0, \quad \beta^4(y+2)G = 0, \\
\text{where} \quad & G = b^2(y-1)^3 - \beta^3z^2(z+y-1).
\end{aligned}$$

*The reader may hold F to be an algebraic field; it is only for completeness that modular fields are included (with proper restrictions).

If $G=0$, set $Y=b(y-1)$, $Z=\beta z$. Then $bG=0$ becomes

$$Y^3 - \beta YZ^2 - bZ^3 = 0.$$

But (10) is irreducible. Hence $Y=Z=0$, so that $y=1, z=0$. Thus $D=\beta, d=0$. Now $L : A = -\beta : 1$. Thus

$$0 = L + \beta A = -8\beta(\delta - b)^2.$$

Hence $\delta=b$. Hence relations (3) reduce to (2), so that *the algebra is a field*.

Let next $G \neq 0$. Then $z=-1, y=-2, x=-8$, by (13) and (12). Thus $\delta=-8b, D=-2\beta, d=-\beta^2$ by (11). The remaining conditions are seen to be satisfied. Indeed, the Hessian now equals

$$(32\beta^3 - 216b^2) \Delta.$$

The multiplication table of the resulting algebra is given by (1) and

$$(14) \quad i^2=j, \quad ij=ji=b+\beta i, \quad j^2=-\beta^2-8bi-2\beta j.$$

A direct proof that division is always uniquely possible in this remarkable algebra is given in the papers cited above.

Consider next the case $\beta=0$. Eliminating $d\delta$ between (7) and (8), we get $b^2D^3-d^3=0$. If $d \neq 0$, set $x=bD/d$. Then $x^3-b=0$, contrary to the irreducibility of (10). Hence $d=0, D=0$. Set $\lambda=-(\delta+2b)$. Then

$$\Delta = r^3 + rst\lambda + s^3b + t^3b\delta, \quad H = -6r^3\lambda^2 - 6t^3\lambda^2b\delta - 6s^3\lambda^2b + rst(2\lambda^3 + 6^3b^2\delta).$$

Hence H is a constant times Δ if and only if

$$2\lambda^3 + 6^3b^2\delta = -6\lambda^3,$$

viz., if $(\delta-b)^2(\delta+8b)=0$. The resulting algebras are again (2) and (14).

For completeness, we treat the excluded case* of a field with modulus 3. Let first $\beta \neq 0$. From $B : A = b : 1$,

$$(15) \quad b(\beta - D)^2 = d(\delta + 2b).$$

Reducing (9) modulo 3 and eliminating $\delta + 2b$ by (15), we get

$$(\beta - D)[d^2 - \beta d^2(\beta - D) + b^2(\beta - D)^3] = 0.$$

If the second factor vanished, (10) would have a rational root

$$x = -b(\beta - D)/d.$$

*For modulus 2, $H \equiv 0$. A discussion shows that the algebra is a field.

Hence $\beta=D$. Hence $A=-\beta^3$, $L=\beta^4+\beta(\delta+2b)^2$. Hence $L:A=-\beta:1$ gives $\delta\equiv b \pmod{3}$. Then (8) becomes $d^2(d+\beta^2)\equiv 0$. Hence $d=0$ or $-\beta^2$, giving again algebras (2) and (14). Finally, let $\beta=0$. Then $A\equiv 0$, so that the Hessian must be identically zero. From C, K , we get

$$bdD(\delta+2b)+b^2D^3+d^3=0, \quad d^2D=d(\delta+2b)^2.$$

If $d\neq 0$, we eliminate $\delta+2b$ and get $(b^2D^3-d^3)^2=0$. Hence $(bDd^{-1})^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Then $H=(\delta+2b)^3rst$, so that $\delta\equiv b$. The algebra is again the field algebra.

In summary, *the only algebras with the required properties are the obvious field-algebra and the remarkable non-field algebra (14).*

The University of Chicago, September, 1906.

THE DIVISION OF ANGLES INTO n EQUAL PARTS.

By J. SAMSONOFF, New York City.

Principle: If by the method given later one has constructed the geometrical loci of vertices of triangles with base XA , whose remote angle at the base is equal respectively to one, two, three, ..., $(n-1)$ times the angle at the vertex of the respective triangle, any angle $CA Y$ is geometrically divisible by 2, 3, ..., n .

Given: Base XA and curves $a, b, c, d, \dots$, the geometrical loci of vertices of triangles, where the remote

$$\begin{aligned} \angle aXA &= \angle XaA, \quad \angle bXA = 2\angle XbA, \quad \dots, \\ \angle eXA &= (n-1)\angle XeA. \end{aligned}$$

To prove: $\angle CA Y$ is geometrically divisible into 2, 3, 4, ..., n equal parts.

Proof: (1) In $\triangle XaA$, $\angle aXA = \angle XaA$ (by hypothesis), $\angle CA Y = \angle aXA + \angle XaA$ (the exterior angle of $\triangle aXA$). Therefore the measure of $\angle XaA$ must be equal to $\frac{1}{2}$ of the measure of $\angle CA Y$.

(2) In $\triangle XbA$, $\angle bXA = 2\angle XbA$ (by hypothesis), $\angle CA Y = \angle bXA + \angle XbA$ (the exterior angle of $\triangle XbA$). Therefore the measure of $\angle XbA$ must be equal to $\frac{1}{3}$ of the measure of $\angle CA Y$.

(n) In $\triangle XeA$, $\angle eXA = (n-1)\angle XeA$ (by hypothesis), $\angle CA Y = \angle eXA + \angle XeA$ (the exterior angle of $\triangle eXA$). Therefore the measure of $\angle XeA$ must be equal to $1/n$ of the measure of $\angle CA Y$.

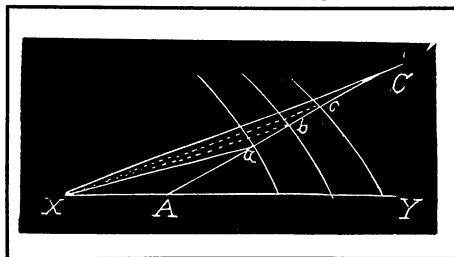


Fig. 1.